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New Lyapunov–Krasovskii Functional for Mixed-Delay-Dependent Stability of Uncertain Linear Neutral Systems

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Abstract The robust stability in a class of uncertain linear neutral systems with time-varying delays is studied. Through choosing multiple integral Lyapunov terms and using some novel integral inequalities, a much tighter estimation on derivative of Lyapunov–Krasovskii (L–K) functional is presented and two stability criteria are expressed in terms of linear matrix inequalities, in which those previously ignored information can be considered. In particular, the proposed Lyapunov technique can effectively consider the interconnection between neutral delay and state one. Finally, two numerical examples with comparing results can show the application area and benefits of the proposed conditions.

Keywords Linear neutral systems · Robust stability · Multiple integral Lyapunov functional · Mixed-delay dependence · Partial element equivalent circuit (PEEC)

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1 Introduction

In past decade, due to the existence of time-delay in many engineering fields, the research on time-delay systems has become a focused topic of theoretical and practical importance. In a large number of control systems, it is well known that the presence of time-delay often leads to the oscillation, instability, or other poor performances [9,10,35,39,47]. Therefore, the stability in various time-delay systems has been widely studied, see [1–8,11–34,36–38,40–46,48–50] for the references and therein. In particular, since neutral delay-differential system involves the time-delay in both state and its derivative and can include many practical models as its special case, the issue on its stability has drawn considerable attention [1–8,11–34,36–38,40–46]. For instance, in [7], by choosing some effective L–K functionals and using linear matrix inequality (LMI) approach, the asymptotical stability in a class of neutral systems with constant time-delay was studied. In [1], a necessary and sufficient condition on exponential stability for time-variable neutral one was derived, in which time-delay was variable. Yet in practice, due to that accurate mathematical model cannot be easily obtained, many works have considered parameter uncertainties in addressed systems [2–4,12,13,16–20,30–33,40,42]. On the one hand, in [3,16,18,33], the LMI criteria on robust stability for uncertain neutral systems have been established. On the other hand, in [2,4,12,13,17,19,20,30–32,40,42], through treating the nonlinearity as system uncertainties, the delay-dependent criteria have been given in terms of LMIs, in which some restricted conditions would be set beforehand. Meanwhile, although the discrete delay can be introduced into communication channels since it is ubiquitous in signal transmission, a system usually has a special nature due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Such an inherent nature can be suitably modeled as the description of distributed delay. Therefore, some works have studied the robust stability in neutral systems with distributed delay [3,18,20,30,31]. Furthermore, through using several delay-partitioning ideas, some less conservative results have been obtained and their conservatism can be greatly reduced by dividing the delay intervals [12,13,31,32]. There also exist some works involving the effect of other factors, such as stochastic disturbance [5,21,27], leakage delays with impulse [2,34], neutral proportional delays [29], H_∞ performance with Markov jumping [43], switching effects [6,22,41,44], Lur'e neutral type [8,23,24,41], together with its application to PEEC model [46]. It is worth pointing out that since the triple Lyapunov technique was put forward in [36], it has received considerable attention and achieved great improvements [11,15,37], which was also employed to study the neutral cases [41].

Since the L–K stability theory was used to address the delay dependence, many effective techniques have been proposed, such as free-weighting matrices, various integral inequalities, convex combination, and delay-partitioning ideas [2,4,12,13,17,19,20,30–32,40,42]. Though these results above are elegant, there still exist some points waiting for the improvements. Firstly, since the results in [21,34,45] were presented in the forms of complicated inequalities, they could not be conveniently checked by resorting to the most developed algorithms and applied to real systems. Secondly, after Jesen integral inequality was widely used, the Wirtinger-based ones were also presented to tackle time-delay system and some useful information could be reconsidered [37], which had been ignored by the Jesen one. Furthermore, the works [11,37,48,49] proposed some novel free-matrix-based integral inequalities combining the features of free-weighting matrices and integral one, in which some slack matrices would induce computation complexity. Later, the inequalities in [11,48] were also extended to discrete-time case [49]. It is worth noting that those techniques in [11, 37,48,49] aimed to single integral form. Because double integral form was derived from triple integral ones, it also has been utilized to reduce the conservatism [28,29]. In particular, the works [14,28,29] introduced some novel Wirtinger-based double integral inequalities and auxiliary function-based ones, and they could give much tighter bound on the double integral forms. Yet except for the works [49,50], the techniques in [11,14,28,29,37,48,49] were employed to tackle constant time-delay. Since most practical cases are concerning about time-varying ones, some new problems will be unavoidably encountered. Thirdly, though there always exist multiple time-delays in neutral system, most existent works individually employed the information of each time-delay to choose L–K functional [2,4,12,13,16–20,30–33,40,42]. Yet, in practical cases, the neutral delay is always different from the state one, and few works have utilized their interconnected relationship to achieve stability results. Though the works [3,7] have given some preliminary discussions, there still exists much room on this point. In particular, in [3,7], the proposed Lyapunov terms seemed to be simple and they could not effectively represent the interconnection between the neutral delay and state one. Therefore, some novel techniques need to be put forward.

In this work, the robust stability for a class of uncertain time-delay neutral systems will be deeply studied. Together with the interconnected relationship between the time-delays in the studied system, an improved Lyapunov–Krasovskii functional involving some multiple integral terms will be constructed and several novel integral inequalities will be utilized to give much tighter upper bound on L–K functional's derivative. The derived criteria are presented in terms of LMIs, and they can be easily tested. Finally, two numerical examples will demonstrate the reduced conservatism and application of the derived results.

Notations The term L–K functional denotes the abbreviation of Lyapunov–Krasovskii functional; the set $\mathbf{R}^n$ denotes the n-dimensional Euclidean space and $\mathbf{R}^{n\times m}$ is the set of $n \times m$ real matrices; I denotes an identity matrix of appropriate dimension; $\mathbf{sym}\{\mathrm{X}\}$ means the sum of X and its symmetric matrix, i.e., $\mathbf{sym}\{X\} = X + X^{\mathrm{T}}$; and the symmetric term in a symmetric matrix is denoted by $*$, i.e., $\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ * & Z \end{bmatrix}$.

2 Model Descriptions and Preliminaries

In this work, we consider the uncertain neutral systems with time-varying delays as

$$\begin{aligned} &\dot{x}(t) - [C + \Delta C(t)]\dot{x}(t - \tau_1(t)) = [A + \Delta A(t)]x(t) \\ &\qquad + [B + \Delta B(t)]x(t - \tau_2(t)), \quad t \geq t_0; \\ &x(t) = \phi(t), \quad t \leq t_0, \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state of system (1) and A, B, C are the constant matrices of appropriate dimensions with $\| C + \Delta C(t) \| < 1$.

The following assumptions on the system (1) are made throughout this paper.

H1 The functions $\tau_i(t)$ $(i = 1, 2)$ denote the time-varying delays and satisfy

$$0 \le \tau_i(t) \le \tau_i, \quad \upsilon_i \le \dot{\tau}_i(t) \le \mu_i \quad (i = 1, 2), \tag{2}$$

where $\tau_i, \upsilon_i, \mu_i$ $(i = 1, 2)$ are scalars.

H2 The uncertainties $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ satisfy the following conditions

$$\begin{aligned} &\big[\Delta A(t)\ \Delta B(t)\ \Delta C(t)\big] = F\Delta(t)\big[E_1\ E_2\ E_3\big]; \\ &\Delta(t) = \Lambda(t)[I - J\Lambda(t)]^{-1}, \quad I - J^{\mathrm{T}}J > 0, \end{aligned} \tag{3}$$

in which F, J, E_i $(i = 1, 2, 3)$ are known constant matrices of appropriate dimensions and $\Lambda(t)$ is an unknown time-varying matrix satisfying $\Lambda^{\mathrm{T}}(t)\Lambda(t) \le I$.

Remark 1 In **H1**, on the one hand, when time-delays $\tau_i(t)$ $(i = 1, 2)$ are constant, one can easily check that $\upsilon_i = \mu_i = 0$ $(i = 1, 2)$ and , on the other hand, when time-delays $\tau_i(t)$ $(i = 1, 2)$ are time variable, it is easy to check that the values of υ_i $(i = 1, 2)$ have to be less than 0 and the ones of μ_i $(i = 1, 2)$ have to be greater than 0, which guarantees $\tau_i(t)$ $(i = 1, 2)$ to be variable and bounded in $[0, \tau_i]$ $(i = 1, 2)$. Yet many present works aimed to study the upper bound of $\dot{\tau}_i(t)$ $(i = 1, 2)$, but neglected the information of its lower bound, which would unavoidably result in some conservatism.

3 Delay-Dependent Stability Criteria

In what follows, some denotations will be given to simplify the proof procedure

$$\begin{aligned} &\bar{\tau}_i(t) = \tau_i - \tau_i(t),\ \bar{\mu}_i = \mu_i - \upsilon_i\ (i = 1, 2),\ \tau_{21} = \tau_2 - \tau_1, \\ &\delta_{21} = \tau_2^2 - \tau_1^2,\ \theta_{21} = \tau_2^3 - \tau_1^3; \end{aligned} \tag{4}$$

$$\varphi_i(t) = \frac{1}{\tau_i(t)} \int_{t-\tau_i(t)}^{t} x(s)\mathrm{d}s, \quad \nu_i(t) = \frac{2}{\tau_i^2(t)} \int_{t-\tau_i(t)}^{t} \int_{t-\tau_i(t)}^{s} x(u)\mathrm{d}u\mathrm{d}s \ (i = 1, 2); \tag{5}$$

$$\varrho_i(t) = \frac{1}{\bar{\tau}_i(t)} \int_{t-\tau_i}^{t-\tau_i(t)} x(s)\mathrm{d}s, \quad \omega_i(t) = \frac{2}{\bar{\tau}_i^2(t)} \int_{t-\tau_i}^{t-\tau_i(t)} \int_{t-\tau_i}^{s} x(u)\mathrm{d}u\mathrm{d}s \ (i = 1, 2); \tag{6}$$

$$\begin{aligned} &\alpha(t) = \frac{1}{\tau_{21}} \int_{t-\tau_2}^{t-\tau_1} x(s)\mathrm{d}s, \quad \beta(t) = \frac{2}{\tau_{21}^2} \int_{t-\tau_2}^{t-\tau_1} \int_{t-\tau_2}^{s} x(u)\mathrm{d}u\mathrm{d}s, \\ &\gamma(t) = \frac{2}{\delta_{21}} \int_{-\tau_2}^{-\tau_1} \int_{t+s}^{t} x(u)\mathrm{d}u\mathrm{d}s; \end{aligned} \tag{7}$$

$$e_i^{\mathrm{T}} = \big[\, 0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(19-i)n} \,\big] \ (1 \le i \le 19); \tag{8}$$

$$\Lambda = \Big[e_1 \quad e_4 \quad 0_{19n\cdot 2n} \quad \tau_{21}e_{17}\Big], \quad \$ = \Big[e_{14} \quad e_{16} \quad e_1 - e_4 \quad e_1 - e_5 \quad e_4 - e_5\Big]; \tag{9}$$

$$\Phi_i = \left[0_{19n\cdot(i+1)n} \quad e_{5+i} \quad 0_{19n\cdot(3-i)n} \right] (i = 1, 2),$$
$$\Psi_j = \left[0_{19n\cdot(j+1)n} \quad e_{7+j} \quad 0_{19n\cdot(3-j)n} \right] (j = 1, 2); \tag{10}$$
$$E_{1i} = \begin{bmatrix} e_1 - e_{i+1} \\ e_1 + e_{i+1} - 2e_{i+5} \\ e_1 - e_{i+1} + 6e_{i+5} - 12e_{i+9} \end{bmatrix} (i = 1, 2),$$
$$E_{2j} = \begin{bmatrix} e_{j+1} - e_{j+3} \\ e_{j+1} + e_{j+3} - 2e_{j+7} \\ e_{j+1} - e_{j+3} + 6e_{j+7} - 12e_{j+11} \end{bmatrix} (j = 1, 2); \tag{11}$$
$$\begin{aligned} \eta^{\mathrm{T}}(t) = \Big[& x^{\mathrm{T}}(t) \;\; x^{\mathrm{T}}(t - \tau_1(t)) \;\; x^{\mathrm{T}}(t - \tau_2(t)) \;\; x^{\mathrm{T}}(t - \tau_1) \\ & x^{\mathrm{T}}(t - \tau_2) \;\; \varphi_1^{\mathrm{T}}(t) \;\; \varphi_2^{\mathrm{T}}(t) \;\; \varrho_1^{\mathrm{T}}(t) \;\; \varrho_2^{\mathrm{T}}(t) \;\; \nu_1^{\mathrm{T}}(t) \\ & \nu_2^{\mathrm{T}}(t) \;\; \omega_1^{\mathrm{T}}(t) \;\; \omega_2^{\mathrm{T}}(t) \;\; \dot{x}^{\mathrm{T}}(t) \;\; \dot{x}^{\mathrm{T}}(t - \tau_1(t)) \\ & \dot{x}^{\mathrm{T}}(t - \tau_1) \;\; \alpha^{\mathrm{T}}(t) \;\; \beta^{\mathrm{T}}(t) \;\; \gamma^{\mathrm{T}}(t) \Big]. \end{aligned} \tag{12}$$

Now we will give one novel delay-dependent stability criterion on the nominal system of (1).

Theorem 1 *For any given scalars* $\tau_i \geq 0, \mu_i, \upsilon_i, \bar{\mu}_i$ $(i = 1, 2), \tau_{21}, \delta_{21}, \theta_{21}$ *in* **H1**, *the nominal system of* (1) *is asymptotically stable, if there exist* $5n \times 5n$ *matrix* $P > 0$, $n \times n$ *matrices* $Q_i > 0$ $(i = 1, \ldots, 6)$, $X_i > 0, Y_i > 0, Z_i > 0, W_i > 0$ $(i = 1, 2)$, $U > 0, V > 0, W > 0, X > 0, Y > 0, N_i$ $(i = 1, 2, 3, 4)$, *and* $3n \times 3n$ *constant matrices* $\bar{X}_i = \mathrm{diag}\{X_i, 3X_i, 5X_i\}$, $3n \times n$ *matrices* U_i *satisfying* $\begin{bmatrix} \bar{X}_i & U_i \\ * & \bar{X}_i \end{bmatrix} \geq 0$ $(i = 1, 2)$ *such that as for* $g \in \{1, 2\};\ h \in \{5, 6\};\ i, j \in \{1, 2\}$, *the LMIs in* (13)–(14) *hold*

$$\begin{bmatrix} \mathbf{\Omega} + \mathbf{\Delta} + \mathbf{\Xi}(g, h) + \tau_i \mathbf{\Upsilon}_{i2} + \tau_j \mathbf{\Upsilon}_{j1} & E_{2i}^{\mathrm{T}} U_i \\ * & -\tau_i \bar{X}_i \end{bmatrix} < 0, \tag{13}$$
$$\begin{bmatrix} \mathbf{\Omega} + \mathbf{\Delta} + \mathbf{\Xi}(g, h) + \tau_i \mathbf{\Upsilon}_{i2} + \tau_j \mathbf{\Upsilon}_{j1} & E_{1j}^{\mathrm{T}} U_j \\ * & -\tau_j \bar{X}_j \end{bmatrix} < 0, \tag{14}$$

where the terms E_{2i}, E_{1j} are defined in (11), $\mathbf{\Xi}(g, h) = \bar{\mu}_1 \left[e_2^{\mathrm{T}} Q_g e_2 + e_{15}^{\mathrm{T}} Q_{g+2} e_{15} \right] + \bar{\mu}_2 e_3^{\mathrm{T}} Q_h e_3$, and

$$\begin{aligned} \mathbf{\Delta} = \mathbf{sym}\left\{ \Lambda P \$^{\mathrm{T}} \right\} & - \sum_{i=1}^{2} \left\{ \frac{1}{\tau_i} \begin{bmatrix} E_{1i} \\ E_{2i} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{X}_i & U_i \\ * & \bar{X}_i \end{bmatrix} \begin{bmatrix} E_{1i} \\ E_{2i} \end{bmatrix} \right. \\ & + 2 \left[\frac{e_1}{2} + e_{5+i} - \frac{3e_{9+i}}{2} \right]^{\mathrm{T}} (Z_1 + W_1) \\ & \times \left[\frac{e_1}{2} + e_{5+i} - \frac{3e_{9+i}}{2} \right] + \left[e_{5+i} - e_{1+i} \right]^{\mathrm{T}} Z_i \left[e_{5+i} - e_{1+i} \right] \end{aligned}$$

$$
\begin{aligned}
&+2\Big[\frac{e_{1+i}}{2}+e_{7+i}-\frac{3e_{11+i}}{2}\Big]^{\mathrm{T}}(Z_2+W_2)\Big[\frac{e_{1+i}}{2}+e_{7+i}-\frac{3e_{11+i}}{2}\Big]\\
&+\big[e_{7+i}-e_{3+i}\big]^{\mathrm{T}}Z_i\big[e_{7+i}-e_{3+i}\big]\\
&+\big[e_1-e_{5+i}\big]^{\mathrm{T}}W_i\big[e_1-e_{5+i}\big]+\big[e_{1+i}-e_{7+i}\big]^{\mathrm{T}}W_i\big[e_{1+i}-e_{7+i}\big]\Big\}\\
&-\big[e_4+e_5-2e_{17}\big]^{\mathrm{T}}(3V)\big[e_4+e_5-2e_{17}\big]\\
&-\big[e_4-e_5+6e_{17}-6e_{18}\big]^{\mathrm{T}}(5V)\big[e_4-e_5+6e_{17}-6e_{18}\big],
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Upsilon}_{i1}&=\mathbf{sym}\big\{\Phi_i P\$^{\mathrm{T}}\big\}-e_{i+5}^{\mathrm{T}}Y_ie_{i+5}-3(e_{i+5}-e_{i+9})^{\mathrm{T}}Y_i(e_{i+5}-e_{i+9})\\
&\quad-(e_{i+1}-e_{i+3})^{\mathrm{T}}\frac{Z_i}{2\tau_i}(e_{i+1}-e_{i+3})-\frac{1}{\tau_i^2}E_{2i}^{\mathrm{T}}X_iE_{2i},\\
\boldsymbol{\Upsilon}_{j2}&=\mathbf{sym}\big\{\Psi_j P\$^{\mathrm{T}}\big\}-e_{j+7}^{\mathrm{T}}Y_je_{j+7}-3(e_{j+7}-e_{j+11})^{\mathrm{T}}Y_j(e_{j+7}-e_{j+11})\\
&\quad-(e_1-e_{j+1})^{\mathrm{T}}\frac{W_j}{2\tau_j}(e_1-e_{j+1})-\frac{1}{\tau_j^2}E_{1j}^{\mathrm{T}}X_jE_{1j}
\end{aligned}
$$

with e_i $(1\le i\le 19)$ expressed in (8) and part elements of matrix $\boldsymbol{\Omega}=\big[\boldsymbol{\Omega}_{ij}\big]_{19n\times 19n}$ listed as

$$
\begin{aligned}
\boldsymbol{\Omega}_{11}&=\tau_1Y_1+\tau_2Y_2+Q_1+Q_5+\tau_{21}^2W+N_1^{\mathrm{T}}A+A^{\mathrm{T}}N_1-\tau_{21}^2X-0.25\delta_{21}^2Y,\\
\boldsymbol{\Omega}_{22}&=(\upsilon_1-1)Q_1+(1-\mu_1)Q_2,\\
\boldsymbol{\Omega}_{33}&=(\upsilon_2-1)Q_5+(1-\mu_2)Q_6+N_4^{\mathrm{T}}B+B^{\mathrm{T}}N_4,\\
\boldsymbol{\Omega}_{44}&=\tau_{21}U-Q_2-V,\ \ \boldsymbol{\Omega}_{45}=V,\ \ \boldsymbol{\Omega}_{55}=\tau_{21}U-Q_6-V,\\
\boldsymbol{\Omega}_{14,14}&=\tau_1X_1+\tau_2X_2+0.25(\tau_1^2Z_1+\tau_2^2Z_2+\tau_1^2W_1+\tau_2^2W_2+\delta_{21}^2X)\\
&\quad+Q_3+\tau_{21}^2V-N_2-N_2^{\mathrm{T}}+\frac{\theta_{21}^2}{36}Y,\\
\boldsymbol{\Omega}_{15,15}&=N_3^{\mathrm{T}}C+C^{\mathrm{T}}N_3+(\upsilon_1-1)Q_3+(1-\mu_1)Q_4,\ \ \boldsymbol{\Omega}_{16,16}=-Q_4,\\
\boldsymbol{\Omega}_{17,17}&=-\tau_{21}^2(4W+X),\ \ \boldsymbol{\Omega}_{18,18}=-3\tau_{21}^2W,\\
\boldsymbol{\Omega}_{19,19}&=-0.25\delta_{21}^2Y;\ \ \boldsymbol{\Omega}_{13}=N_1^{\mathrm{T}}B+A^{\mathrm{T}}N_4,\\
\boldsymbol{\Omega}_{1,14}&=A^{\mathrm{T}}N_2-N_1^{\mathrm{T}},\ \ \boldsymbol{\Omega}_{1,15}=N_1^{\mathrm{T}}C+A^{\mathrm{T}}N_3,\\
\boldsymbol{\Omega}_{1,17}&=\tau_{21}^2X,\ \ \boldsymbol{\Omega}_{1,19}=0.25\delta_{21}^2Y,\\
\boldsymbol{\Omega}_{3,14}&=B^{\mathrm{T}}N_2-N_4^{\mathrm{T}},\ \ \boldsymbol{\Omega}_{3,15}=B^{\mathrm{T}}N_3+N_4^{\mathrm{T}}C,\\
\boldsymbol{\Omega}_{14,15}&=N_2^{\mathrm{T}}C-N_3,\ \ \boldsymbol{\Omega}_{17,18}=3\tau_{21}^2W.
\end{aligned}
$$

Proof Now through setting $\zeta^{\mathrm{T}}(t)=\Big[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau_1)\ \int_{t-\tau_1}^{t}x^{\mathrm{T}}(s)\mathrm{d}s\ \int_{t-\tau_2}^{t}x^{\mathrm{T}}(s)\mathrm{d}s\ \int_{t-\tau_2}^{t-\tau_1}x^{\mathrm{T}}(s)\mathrm{d}s\Big]$ and using the assumption **H1**, we can construct the Lyapunov–Krasovskii functional as

$$
V(x_t)\ =\ V_1(x_t)+V_2(x_t)+V_3(x_t)+V_4(x_t), \tag{15}
$$

where

$$
\begin{aligned}
V_1(x_t) &= \zeta^{\mathrm{T}}(t)P\zeta(t),\\
V_2(x_t) &= \int_{t-\tau_1(t)}^{t}\left[x^{\mathrm{T}}(s)Q_1x(s)+\dot{x}^{\mathrm{T}}(s)Q_3\dot{x}(s)\right]\mathrm{d}s\\
&\quad+\int_{t-\tau_1}^{t-\tau_1(t)}\left[x^{\mathrm{T}}(s)Q_2x(s)+\dot{x}^{\mathrm{T}}(s)Q_4\dot{x}(s)\right]\mathrm{d}s\\
&\quad+\int_{t-\tau_2(t)}^{t}x^{\mathrm{T}}(s)Q_5x(s)\mathrm{d}s+\int_{t-\tau_2}^{t-\tau_2(t)}x^{\mathrm{T}}(s)Q_6x(s)\mathrm{d}s,\\
V_3(x_t) &= \sum_{i=1}^{2}\left\{\int_{-\tau_i}^{0}\int_{t+s}^{t}\left[\dot{x}^{\mathrm{T}}(\theta)X_i\dot{x}(\theta)+x^{\mathrm{T}}(\theta)Y_ix(\theta)\right]\mathrm{d}\theta\mathrm{d}s\right.\\
&\quad+\frac{1}{2}\int_{t-\tau_i}^{t}\int_{t-\tau_i}^{\varrho}\int_{t-\tau_i}^{\theta}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho\\
&\quad\left.+\frac{1}{2}\int_{t-\tau_i}^{t}\int_{\varrho}^{t}\int_{\theta}^{t}\dot{x}^{\mathrm{T}}(s)W_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho\right\},\\
V_4(x_t) &= \tau_{21}\int_{t-\tau_2}^{t-\tau_1}x^{\mathrm{T}}(s)Ux(s)\mathrm{d}s+\tau_{21}\int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t}\left[\dot{x}^{\mathrm{T}}(\theta)V\dot{x}(\theta)\right.\\
&\quad\left.+x^{\mathrm{T}}(\theta)Wx(\theta)\right]\mathrm{d}\theta\mathrm{d}s\\
&\quad+\frac{\delta_{21}}{2}\int_{-\tau_2}^{-\tau_1}\int_{\varrho}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho\\
&\quad+\frac{\theta_{21}}{6}\int_{-\tau_2}^{-\tau_1}\int_{\mu}^{0}\int_{\varrho}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Y\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho\mathrm{d}\mu.
\end{aligned}
$$

Firstly, we can easily check that for $i=1,2$,

$$
\begin{aligned}
\int_{t-\tau_i}^{t}x(s)\mathrm{d}s &= \frac{\tau_i(t)}{\tau_i(t)}\int_{t-\tau_i(t)}^{t}x(s)\mathrm{d}s+\frac{\bar{\tau}_i(t)}{\bar{\tau}_i(t)}\int_{t-\tau_i}^{t-\tau_i(t)}x(s)\mathrm{d}s\\
&= \tau_i(t)\varphi_i(t)+\bar{\tau}_i(t)\varrho_i(t).
\end{aligned}\tag{16}
$$

Then, together with the denotations in (4)–(12), the derivative of $V_i(x_t)$ $(i=1,2,3)$ along the nominal system of (1) can be directly computed out as

$$
\begin{aligned}
\dot{V}_1(x_t) &= 2\zeta^{\mathrm{T}}(t)P\dot{\zeta}(t)\\
&= 2\eta^{\mathrm{T}}(t)\Big[e_1\ \ e_4\ \ \tau_1(t)e_6+\bar{\tau}_1(t)e_8\quad \tau_2(t)e_7+\bar{\tau}_2(t)e_9\quad \tau_{21}e_{17}\Big]\\
&\qquad\times P\Big[e_{14}\ \ e_{16}\ \ e_1-e_4\ \ e_1-e_5\ \ e_4-e_5\Big]^{\mathrm{T}}\eta(t)\\
&= 2\eta^{\mathrm{T}}(t)\Big\{\Big[e_1\ \ e_4\ \ 0_{19n\cdot 2n}\ \ \tau_{21}e_{17}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{2} \tau_i(t)\Big[0_{19n\cdot(i+1)n} \;\; e_{5+i} \;\; 0_{19n\cdot(3-i)n}\Big] \\
&\quad + \sum_{j=1}^{2} \bar{\tau}_j(t)\Big[0_{19n\cdot(j+1)n} \;\; e_{7+j} \;\; 0_{19n\cdot(3-j)n}\Big]\Big\} \\
&\quad \times P\Big[e_{14} \;\; e_{16} \;\; e_1 - e_4 \;\; e_1 - e_5 \;\; e_4 - e_5\Big]^{\mathrm{T}} \eta(t) \\
&= \eta^{\mathrm{T}}(t)\mathbf{sym}\Big\{\Lambda P\$^{\mathrm{T}} + \sum_{i=1}^{2} \tau_i(t)\Phi_i P\$^{\mathrm{T}} + \sum_{j=1}^{2} \bar{\tau}_j(t)\Psi_j P\$^{\mathrm{T}}\Big\}\eta(t); \quad (17)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_2(x_t) &= \left[x^{\mathrm{T}}(t)(Q_1 + Q_5)x(t) + \dot{x}^{\mathrm{T}}(t)Q_3\dot{x}(t)\right] \\
&\quad - \left[x^{\mathrm{T}}(t-\tau_1)Q_2 x(t-\tau_1) + \dot{x}^{\mathrm{T}}(t-\tau_1)Q_4\dot{x}(t-\tau_1)\right] \\
&\quad - [1 - \dot{\tau}_1(t)]\Big[x^{\mathrm{T}}(t-\tau_1(t))(Q_1 - Q_2)x(t-\tau_1(t)) \\
&\quad + \dot{x}^{\mathrm{T}}(t-\tau_1(t))(Q_3 - Q_4)\dot{x}(t-\tau_1(t))\Big] \\
&\quad - [1 - \dot{\tau}_2(t)]x^{\mathrm{T}}(t-\tau_2(t))(Q_5 - Q_6)x(t-\tau_2(t)) \\
&\quad - x^{\mathrm{T}}(t-\tau_2)Q_6 x(t-\tau_2); \quad (18)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_3(x_t) &= \sum_{l=1}^{2} \Big\{\tau_i\left[\dot{x}^{\mathrm{T}}(t)X_i\dot{x}(t) + x^{\mathrm{T}}(t)Y_i x(t)\right] \\
&\quad + \frac{\tau_i^2}{4}\dot{x}^{\mathrm{T}}(t)\big(Z_i + W_i\big)\dot{x}(t) - \int_{t-\tau_i}^{t} \left[\dot{x}^{\mathrm{T}}(\theta)X_i\dot{x}(\theta)\right. \\
&\quad \left. + x^{\mathrm{T}}(\theta)Y_i x(\theta)\right]\mathrm{d}\theta - \frac{1}{2}\int_{t-\tau_i}^{t}\int_{t-\tau_i}^{\theta} \dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\
&\quad - \frac{1}{2}\int_{t-\tau_i}^{t}\int_{\theta}^{t} \dot{x}^{\mathrm{T}}(s)W_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\Big\}. \quad (19)
\end{aligned}
$$

Now we use Lemmas 1–3 and the denotations (5)–(12) to compute out the estimations on integral terms in (19). Firstly, since $\bar{X}_i = \mathrm{diag}\{X_i, 3X_i, 5X_i\}$ and $\begin{bmatrix} \bar{X}_i & U_i \\ * & \bar{X}_i \end{bmatrix} \geq 0$, it follows from Lemma 1 that

$$
\begin{aligned}
-\int_{t-\tau_i}^{t} \dot{x}^{\mathrm{T}}(\theta)X_i\dot{x}(\theta)\mathrm{d}\theta &= -\left[\int_{t-\tau_i(t)}^{t} + \int_{t-\tau_i}^{t-\tau_i(t)}\right]\dot{x}^{\mathrm{T}}(\theta)X_i\dot{x}(\theta)\mathrm{d}\theta \\
&\leq -\frac{1}{\tau_i}\eta^{\mathrm{T}}(t)\begin{bmatrix} E_{1i} \\ E_{2i} \end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix} \bar{X}_i & U_i \\ * & \bar{X}_i \end{bmatrix}\right. \\
&\quad \left. + \begin{bmatrix} \frac{\bar{\tau}_i(t)}{\tau_i}T_i & 0 \\ * & \frac{\tau_i(t)}{\tau_i}T_i \end{bmatrix}\right)\begin{bmatrix} E_{1i} \\ E_{2i} \end{bmatrix}\eta(t), \quad (20)
\end{aligned}
$$

where $T_i = \bar{X}_i - U_i \bar{X}_i^{-1} U_i^{\mathrm{T}}$ $(i = 1, 2)$, the terms E_{1i}, E_{2i} are expressed in (11), and $\eta(t)$ is given in (12). In the next, we will employ Lemmas 2–3 to estimate the following integral terms as

$$\begin{aligned}
-\int_{t-\tau_i}^{t} x^{\mathrm{T}}(\theta) Y_i x(\theta) \mathrm{d}\theta \; &= \; -\left[\int_{t-\tau_i(t)}^{t} + \int_{t-\tau_i}^{t-\tau_i(t)}\right] x^{\mathrm{T}}(\theta) Y_i x(\theta) \mathrm{d}\theta \\
&\le -\tau_i(t)\varphi_i^{\mathrm{T}}(t) Y_i \varphi_i(t) - 3\tau_i(t)\big[\varphi_i(t) - \nu_i(t)\big]^{\mathrm{T}} Y_i \big[\varphi_i(t) - \nu_i(t)\big] \\
&\quad -\bar{\tau}_i(t)\varrho_i^{\mathrm{T}}(t) Y_i \varrho_i(t) \\
&\quad -3\bar{\tau}_i(t)\big[\varrho_i(t) - \omega_i(t)\big]^{\mathrm{T}} Y_i \big[\varrho_i(t) - \omega_i(t)\big];
\end{aligned} \tag{21}$$

$$\begin{aligned}
&-\frac{1}{2}\int_{t-\tau_i}^{t}\int_{t-\tau_i}^{\theta} \dot{x}^{\mathrm{T}}(s) Z_i \dot{x}(s) \mathrm{d}s \mathrm{d}\theta \\
&= -\frac{1}{2}\left[\int_{t-\tau_i(t)}^{t}\int_{t-\tau_i}^{t-\tau_i(t)} + \int_{t-\tau_i(t)}^{t}\int_{t-\tau_i(t)}^{\theta} + \int_{t-\tau_i}^{t-\tau_i(t)}\int_{t-\tau_i}^{\theta}\right] \dot{x}^{\mathrm{T}}(s) Z_i \dot{x}(s) \mathrm{d}s \mathrm{d}\theta \\
&\le -\Big[x(t-\tau_i(t)) - x(t-\tau_i)\Big]^{\mathrm{T}} \left[\frac{\tau_i(t) Z_i}{2\tau_i}\right] \Big[x(t-\tau_i(t)) - x(t-\tau_i)\Big] \\
&-\Big[\varphi_i(t) - x(t-\tau_i(t))\Big]^{\mathrm{T}} Z_i \Big[\varphi_i(t) - x(t-\tau_i(t))\Big] \\
&-\Big[\varrho_i(t) - x(t-\tau_i)\Big]^{\mathrm{T}} Z_i \Big[\varrho_i(t) - x(t-\tau_i)\Big] \\
&-\left[\frac{x(t)}{2} + \varphi_i(t) - \frac{3}{2}\nu_i(t)\right]^{T} (2Z_i) \left[\frac{x(t)}{2} + \varphi_i(t) - \frac{3}{2}\nu_i(t)\right] \\
&-\left[\frac{1}{2}x(t-\tau_i(t)) + \varrho_i(t) - \frac{3}{2}\omega_i(t)\right]^{\mathrm{T}} (2Z_i) \left[\frac{1}{2}x(t-\tau_i(t)) + \varrho_i(t) - \frac{3}{2}\omega_i(t)\right];
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-\frac{1}{2}\int_{t-\tau_i}^{t}\int_{\theta}^{t} \dot{x}^{\mathrm{T}}(s) W_i \dot{x}(s) \mathrm{d}s \mathrm{d}\theta \\
&= -\frac{1}{2}\left[\int_{t-\tau_i(t)}^{t}\int_{\theta}^{t} + \int_{t-\tau_i}^{t-\tau_i(t)}\int_{t-\tau_i(t)}^{t} + \int_{t-\tau_i}^{t-\tau_i(t)}\int_{\theta}^{t-\tau_i(t)}\right] \dot{x}^{\mathrm{T}}(s) W_i \dot{x}(s) \mathrm{d}s \mathrm{d}\theta \\
&\le -\Big[x(t) - x(t-\tau_i(t))\Big]^{\mathrm{T}} \left[\frac{\bar{\tau}_i(t) W_i}{2\tau_i}\right] \Big[x(t) - x(t-\tau_i(t))\Big] \\
&\quad -\Big[x(t) - \varphi_i(t)\Big]^{\mathrm{T}} W_i \Big[x(t) - \varphi_i(t)\Big] \\
&\quad -\Big[x(t-\tau_i(t)) - \varrho_i(t)\Big]^{\mathrm{T}} W_i \Big[x(t-\tau_i(t)) - \varrho_i(t)\Big] \\
&\quad -\left[\frac{x(t)}{2} + \varphi_i(t) - \frac{3}{2}\nu_i(t)\right]^{T} (2W_i) \left[\frac{x(t)}{2} + \varphi_i(t) - \frac{3}{2}\nu_i(t)\right] \\
&\quad -\left[\frac{1}{2}x(t-\tau_i(t)) + \varrho_i(t) - \frac{3}{2}\omega_i(t)\right]^{\mathrm{T}} (2W_i) \left[\frac{1}{2}x(t-\tau_i(t)) + \varrho_i(t) - \frac{3}{2}\omega_i(t)\right].
\end{aligned} \tag{23}$$

In what follows, we can derive $\dot{V}_4(x_t)$ as

$$\begin{aligned}\dot{V}_4(x_t) = {} & \tau_{21}\Big[x^{\mathrm{T}}(t-\tau_1)Ux(t-\tau_1) - x^{\mathrm{T}}(t-\tau_2)Ux(t-\tau_2)\Big] \\ & + \tau_{21}^2\Big[\dot{x}^{\mathrm{T}}(t)V\dot{x}(t) + x^{\mathrm{T}}(t)Wx(t)\Big] - \tau_{21}\int_{t-\tau_2}^{t-\tau_1}\big[\dot{x}^{\mathrm{T}}(\theta)V\dot{x}(\theta) \\ & + x^{\mathrm{T}}(\theta)Wx(\theta)\big]\mathrm{d}\theta + \dot{x}^{\mathrm{T}}(t)\Big(\frac{\delta_{21}^2}{4}X + \frac{\theta_{21}^2}{36}Y\Big)\dot{x}(t) \\ & - \frac{\delta_{21}}{2}\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\ & - \frac{\theta_{21}}{6}\int_{-\tau_2}^{-\tau_1}\int_{\varrho}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Y\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho. \end{aligned} \tag{24}$$

Now based on Lemmas 2–4, we can, respectively, estimate the integral terms in (24) as

$$\begin{aligned} -\tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot{x}^{\mathrm{T}}(\theta)V\dot{x}(\theta)\mathrm{d}\theta \le & -\Big[x(t-\tau_1)-x(t-\tau_2)\Big]^{\mathrm{T}}V\Big[x(t-\tau_1)-x(t-\tau_2)\Big] \\ & -\Big[x(t-\tau_1)+x(t-\tau_2)-2\alpha(t)\Big]^{\mathrm{T}}(3V)\Big[x(t-\tau_1)+x(t-\tau_2)-2\alpha(t)\Big] \\ & -\Big[x(t-\tau_1)-x(t-\tau_2)+6\alpha(t)-6\beta(t)\Big]^{\mathrm{T}} \\ & \times(5V)\Big[x(t-\tau_1)-x(t-\tau_2)+6\alpha(t)-6\beta(t)\Big]; \end{aligned} \tag{25}$$

$$\begin{aligned} -\tau_{21}\int_{t-\tau_2}^{t-\tau_1}x^{\mathrm{T}}(\theta)Wx(\theta)\mathrm{d}\theta \le & -\tau_{21}^2\alpha^{\mathrm{T}}(t)W\alpha(t) - \tau_{21}^2\Big[\alpha(t)-\beta(t)\Big]^{\mathrm{T}} \\ & \times(3W)\Big[\alpha(t)-\beta(t)\Big]; \end{aligned} \tag{26}$$

$$-\frac{\delta_{21}}{2}\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \le -\tau_{21}^2\Big[x(t)-\alpha(t)\Big]^{\mathrm{T}}X\Big[x(t)-\alpha(t)\Big]; \tag{27}$$

$$-\frac{\theta_{21}}{6}\int_{-\tau_2}^{-\tau_1}\int_{\varrho}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Y\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho \le -\frac{\delta_{21}^2}{4}\Big[x(t)-\gamma(t)\Big]^{\mathrm{T}}Y\Big[x(t)-\gamma(t)\Big]. \tag{28}$$

For any $n \times n$ matrices N_i $(i = 1, 2, 3, 4)$, it follows from the nominal system of (1) that

$$\begin{aligned} 0 = 2\Big[& x^{\mathrm{T}}(t)N_1^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)N_2^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t-\tau_1(t))N_3^{\mathrm{T}} + x^{\mathrm{T}}(t-\tau_2(t))N_4^{\mathrm{T}}\Big]\Big[-\dot{x}(t) \\ & + C\dot{x}(t-\tau_1(t) + Ax(t) + Bx(t-\tau_2(t))\Big]. \end{aligned} \tag{29}$$

Now combining with the terms from (17) to (29), we can verify that $\dot{V}(x_t)$ satisfies

$$
\begin{aligned}
\dot{V}(x_t) \leq{}& \eta^{\mathrm{T}}(t)\Bigg\{\mathbf{\Omega} + \mathbf{sym}\{\Lambda P\$^{\mathrm{T}}\} - \sum_{i=1}^{2}\Bigg\{\frac{1}{\tau_i}\begin{bmatrix} E_{1i} \\ E_{2i}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \bar{X}_i & U_i \\ * & \bar{X}_i\end{bmatrix}\begin{bmatrix} E_{1i} \\ E_{2i}\end{bmatrix} \\
&+ 2\Big[\frac{e_1}{2} + e_{5+i} - \frac{3e_{9+i}}{2}\Big]^{\mathrm{T}}(Z_1 + W_1)\Big[\frac{e_1}{2} + e_{5+i} - \frac{3e_{9+i}}{2}\Big] \\
&+ \big[e_{5+i} - e_{1+i}\big]^{\mathrm{T}} Z_i \big[e_{5+i} - e_{1+i}\big] \\
&+ 2\Big[\frac{e_{1+i}}{2} + e_{7+i} - \frac{3e_{11+i}}{2}\Big]^{\mathrm{T}}(Z_2 + W_2)\Big[\frac{e_{1+i}}{2} + e_{7+i} - \frac{3e_{11+i}}{2}\Big] \\
&+ \big[e_{7+i} - e_{3+i}\big]^{\mathrm{T}} Z_i \big[e_{7+i} - e_{3+i}\big] + \big[e_1 - e_{5+i}\big]^{\mathrm{T}} W_i \big[e_1 - e_{5+i}\big] \\
&+ \big[e_{1+i} - e_{7+i}\big]^{\mathrm{T}} W_i \big[e_{1+i} - e_{7+i}\big]\Bigg\} \\
&- \big[e_4 + e_5 - 2e_{17}\big]^{\mathrm{T}}(3V)\big[e_4 + e_5 - 2e_{17}\big] \\
&- \big[e_4 - e_5 + 6e_{17} - 6e_{18}\big]^{\mathrm{T}}(5V)\big[e_4 - e_5 + 6e_{17} - 6e_{18}\big]\Bigg\}\eta(t) \\
&+ \eta^{\mathrm{T}}(t)\Bigg\{[\dot{\tau}_1(t) - \upsilon_1]\big[e_2^{\mathrm{T}} Q_1 e_2 + e_{15}^{\mathrm{T}} Q_2 e_{15}\big] \\
&+ [\mu_1 - \dot{\tau}_1(t)]\big[e_2^{\mathrm{T}} Q_3 e_2 + e_{15}^{\mathrm{T}} Q_4 e_{15}\big] \\
&+ [\dot{\tau}_2(t) - \upsilon_2] e_3^{\mathrm{T}} Q_5 e_3 + [\mu_2 - \dot{\tau}_2(t)] e_3^{\mathrm{T}} Q_6 e_3 \\
&+ \sum_{i=1}^{2}\tau_i(t)\Big[\mathbf{sym}\{\Phi_i P\$^{\mathrm{T}}\} - (e_{i+1} - e_{i+3})^{\mathrm{T}}\frac{Z_i}{2\tau_i}(e_{i+1} - e_{i+3}) - e_{i+5}^{\mathrm{T}} Y_i e_{i+5} \\
&- 3(e_{i+5} - e_{i+9})^{\mathrm{T}} Y_i (e_{i+5} - e_{i+9}) - \frac{1}{\tau_i^2} E_{2i}^{\mathrm{T}} X_i E_{2i} + \frac{1}{\tau_i^2} E_{2i}^{\mathrm{T}} U_i \bar{X}_i^{-1} U_i^{\mathrm{T}} E_{2i}\Big] \\
&+ \sum_{j=1}^{2}\bar{\tau}_j(t)\Big[\mathbf{sym}\{\Psi_j P\$^{\mathrm{T}}\} - (e_1 - e_{j+1})^{\mathrm{T}}\frac{W_j}{2\tau_j}(e_1 - e_{j+1}) - e_{j+7}^{\mathrm{T}} Y_j e_{j+7} \\
&- 3(e_{j+7} - e_{j+11})^{\mathrm{T}} Y_j (e_{j+7} - e_{j+11}) - \frac{1}{\tau_j^2} E_{1j}^{\mathrm{T}} X_j E_{1j} \\
&+ \frac{1}{\tau_j^2} E_{1j}^{\mathrm{T}} U_j \bar{X}_j^{-1} U_j^{\mathrm{T}} E_{1j}\Big]\Bigg\}\eta(t) \\
={}& \eta^{\mathrm{T}}(t)\big(\mathbf{\Omega}+\mathbf{\Delta}\big)\eta(t) + \eta^{\mathrm{T}}(t)\Bigg\{[\dot{\tau}_1(t) - \upsilon_1]\Big[e_2^{\mathrm{T}} Q_1 e_2 + e_{15}^{\mathrm{T}} Q_2 e_{15}\Big] + [\mu_1 - \dot{\tau}_1(t)] \\
&\times\Big[e_2^{\mathrm{T}} Q_3 e_2 + e_{15}^{\mathrm{T}} Q_4 e_{15}\Big] + [\dot{\tau}_2(t) - \upsilon_2] e_3^{\mathrm{T}} Q_5 e_3 + [\mu_2 - \dot{\tau}_2(t)] e_3^{\mathrm{T}} Q_6 e_3 \\
&+ \sum_{i=1}^{2}\tau_i(t)\Big[\mathbf{\Upsilon}_{i2} + \frac{1}{\tau_i^2} E_{2i}^{\mathrm{T}} U_i \bar{X}_i^{-1} U_i^{\mathrm{T}} E_{2i}\Big] \\
&+ \sum_{j=1}^{2}\bar{\tau}_j(t)\Big[\mathbf{\Upsilon}_{j1} + \frac{1}{\tau_j^2} E_{1j}^{\mathrm{T}} U_j \bar{X}_j^{-1} U_j^{\mathrm{T}} E_{1j}\Big]\Bigg\}\eta(t) \\
\doteq{}& \eta^{\mathrm{T}}(t)\tilde{\mathbf{\Phi}}(t)\eta(t), \qquad (30)
\end{aligned}
$$

where $\mathbf{\Omega}$, $\mathbf{\Delta}$, $\mathbf{\Upsilon}_{i2}$, $\mathbf{\Upsilon}_{j1}$ are presented in (13)–(14).

On the other hand, together with the terms in (13)–(14) and the definition on Schur complement, one can easily check that for $g \in \{1,2\}$, $h \in \{5,6\}$, $i, j \in \{1,2\}$, the LMIs in (13)–(14) guarantee

$$\mathbf{\Omega} + \mathbf{\Delta} + \mathbf{\Xi}(g,h) + \tau_i \mathbf{\Upsilon}_{i2} + \tau_j \mathbf{\Upsilon}_{j1} + \frac{1}{\tau_i} E_{2i}^{\mathrm{T}} U_i \bar{X}_i^{-1} U_i^{\mathrm{T}} E_{2i} < 0, \tag{31}$$

$$\mathbf{\Omega} + \mathbf{\Delta} + \mathbf{\Xi}(g,h) + \tau_i \mathbf{\Upsilon}_{i2} + \tau_j \mathbf{\Upsilon}_{j1} + \frac{1}{\tau_j} E_{1j}^{\mathrm{T}} U_j \bar{X}_j^{-1} U_j^{\mathrm{T}} E_{1j} < 0. \tag{32}$$

Then, it follows from Lemma 5 that the terms in (31)–(32) can guarantee $\tilde{\mathbf{\Phi}}(t) < 0$ in (30) to be true. Therefore, it can be concluded that as the conditions (13)–(14) hold, the nominal system of (1) is asymptotically stable. It completes the proof.

In what follows, we will use Lemma 6 to establish one stability criterion on the system (1).

Theorem 2 *For any given scalars $\tau_i \geq 0, \mu_i, \upsilon_i, \bar{\mu}_i$ $(i = 1,2)$, $\tau_{21}, \delta_{21}, \theta_{21}$ in* **H1**, *and the uncertainties satisfying* **H2**, *the system* (1) *is robustly stable, if there exist positive scalars $\eta_{ghij} > 0$ $(g = 1,2; h = 5,6; i, j = 1,2)$, $5n \times 5n$ matrix $P > 0$, $n \times n$ matrices $Q_i > 0$ $(i = 1, \ldots, 6)$, $X_i > 0, Y_i > 0, Z_i > 0, W_i > 0$ $(i = 1,2)$, $U > 0, V > 0, W > 0, X > 0, Y > 0$, N_i $(i = 1,2,3,4)$, and $3n \times 3n$ constant matrices $\bar{X}_i = \mathrm{diag}\{X_i, 3X_i, 5X_i\}$, $3n \times n$ matrices U_i satisfying $\begin{bmatrix} \bar{X}_i & U_i \\ * & \bar{X}_i \end{bmatrix} \geq 0$ $(i = 1,2)$ such that the LMIs in* (33)–(34) *hold*

$$\begin{bmatrix} \mathbf{\Omega} + \mathbf{\Delta} + \mathbf{\Xi}(g,h) + \tau_i \mathbf{\Upsilon}_{i2} + \tau_j \mathbf{\Upsilon}_{j1} & E_{2i}^{\mathrm{T}} U_i & \eta_{ghij} \bar{\mathbf{\Psi}}_1 & \bar{\mathbf{\Psi}}_2 \\ * & -\tau_i \bar{X}_i & 0 & 0 \\ * & * & -\eta_{ghij} I & \eta_{ghij} J \\ * & * & * & -\eta_{ghij} I \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} \mathbf{\Omega} + \mathbf{\Delta} + \mathbf{\Xi}(g,h) + \tau_i \mathbf{\Upsilon}_{i2} + \tau_j \mathbf{\Upsilon}_{j1} & E_{1j}^{\mathrm{T}} U_j & \eta_{ghij} \bar{\mathbf{\Psi}}_1 & \bar{\mathbf{\Psi}}_2 \\ * & -\tau_j \bar{X}_j & 0 & 0 \\ * & * & -\eta_{ghij} I & \eta_{ghij} J \\ * & * & * & -\eta_{ghij} I \end{bmatrix} < 0,$$
$$\forall\, g = 1,2;\ h = 5,6;\ i, j = 1,2, \tag{34}$$

where $\mathbf{\Omega}$, $\mathbf{\Delta}$, $\mathbf{\Xi}(g,h)$, $\mathbf{\Upsilon}_{i2}$, $\mathbf{\Upsilon}_{j1}$ are identical to the corresponding ones in Theorem 1 and

$$\bar{\mathbf{\Psi}}_1 = \begin{bmatrix} E_1 & 0_{n \cdot n} & E_2 & 0_{n \cdot 11n} & E_3 & 0_{n \cdot 4n} \end{bmatrix}^{\mathrm{T}};$$
$$\bar{\mathbf{\Psi}}_2 = \begin{bmatrix} F^{\mathrm{T}} N_1 & 0_{n \cdot n} & F^{\mathrm{T}} N_4 & 0_{n \cdot 10n} & F^{\mathrm{T}} N_2 & F^{\mathrm{T}} N_3 & 0_{n \cdot 4n} \end{bmatrix}^{\mathrm{T}}.$$

Proof Based on the proof procedure of Theorem 1, replacing the corresponding A, B, C in (31)–(32) with the terms $A(t) = A + F\Delta(t)E_1$, $B(t) = B + F\Delta(t)E_2$,

$C(t) = C + F\Delta(t)E_3$, respectively, we can check that the derived matrix inequalities are equivalent to the following ones

$$\begin{aligned}
&\boldsymbol{\Omega} + \boldsymbol{\Delta} + \boldsymbol{\Xi}(g,h) + \tau_i \boldsymbol{\Upsilon}_{i2} + \tau_j \boldsymbol{\Upsilon}_{j1} + \frac{1}{\tau_i} E_{2i}^{\mathrm{T}} U_i \bar{X}_i^{-1} U_i^{\mathrm{T}} E_{2i} \\
&\quad + \bar{\boldsymbol{\Psi}}_1 \Delta(t) \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}} + \left(\bar{\boldsymbol{\Psi}}_1 \Delta(t) \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}}\right)^{\mathrm{T}} < 0, \qquad (35)\\
&\boldsymbol{\Omega} + \boldsymbol{\Delta} + \boldsymbol{\Xi}(g,h) + \tau_i \boldsymbol{\Upsilon}_{i2} + \tau_j \boldsymbol{\Upsilon}_{j1} + \frac{1}{\tau_j} E_{1j}^{\mathrm{T}} U_j \bar{X}_j^{-1} U_j^{\mathrm{T}} E_{1j} \\
&\quad + \bar{\boldsymbol{\Psi}}_1 \Delta(t) \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}} + \left(\bar{\boldsymbol{\Psi}}_1 \Delta(t) \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}}\right)^{\mathrm{T}} < 0. \qquad (36)
\end{aligned}$$

On the basis of Lemma 6, there must exist some positive scalars $\rho_{ghij} > 0$ $(g, i, j = 1, 2;\ h = 5, 6)$ such that

$$\begin{aligned}
&\boldsymbol{\Omega} + \boldsymbol{\Delta} + \boldsymbol{\Xi}(g,h) + \tau_i \boldsymbol{\Upsilon}_{i2} + \tau_j \boldsymbol{\Upsilon}_{j1} \\
&+ \frac{1}{\tau_i} E_{2i}^{\mathrm{T}} U_i \bar{X}_i^{-1} U_i^{\mathrm{T}} E_{2i} + \begin{bmatrix} \rho_{ghij}^{-1} \bar{\boldsymbol{\Psi}}_1^{\mathrm{T}} \\ \rho_{ghij} \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & -J \\ -J^{\mathrm{T}} & I \end{bmatrix}^{-1} \begin{bmatrix} \rho_{ghij}^{-1} \bar{\boldsymbol{\Psi}}_1^{\mathrm{T}} \\ \rho_{ghij} \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}} \end{bmatrix} < 0; \qquad (37)\\
&\boldsymbol{\Omega} + \boldsymbol{\Delta} + \boldsymbol{\Xi}(g,h) + \tau_i \boldsymbol{\Upsilon}_{i2} + \tau_j \boldsymbol{\Upsilon}_{j1} \\
&+ \frac{1}{\tau_j} E_{1j}^{\mathrm{T}} U_j \bar{X}_j^{-1} U_j^{\mathrm{T}} E_{1j} + \begin{bmatrix} \rho_{ghij}^{-1} \bar{\boldsymbol{\Psi}}_1^{\mathrm{T}} \\ \rho_{ghij} \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & -J \\ -J^{\mathrm{T}} & I \end{bmatrix}^{-1} \begin{bmatrix} \rho_{ghij}^{-1} \bar{\boldsymbol{\Psi}}_1^{\mathrm{T}} \\ \rho_{ghij} \bar{\boldsymbol{\Psi}}_2^{\mathrm{T}} \end{bmatrix} < 0. \qquad (38)
\end{aligned}$$

Then, together with the definition on Schur complement, the inequalities in (37)–(38) are equivalent to the LMIs in (33)–(34) by setting $\eta_{ghij} = \rho_{ghij}^{-2}$. It completes the proof.

Remark 2 In Theorems 1–2, the L–K functional terms in $V(x_t)$ have effectively utilized the information of the neutral delay and state one, and some novel techniques have been used, which can help reduce the conservatism more efficiently than ever. Two sufficient conditions are presented, and it is convenient to check their feasibility without tuning any parameters by resorting to the LMI in MATLAB Toolbox.

Remark 3 Based on comparing discussions, these integral inequalities proposed in [11,28,29,37,48–50] can help to extend the application area efficiently. Therefore, together with Lemmas 1–5, our work has used and improved those novel Wirtinger-based integral inequalities and auxiliary function-based ones to tackle the multiple integral Lyapunov terms and those ignored information has been reconsidered.

Remark 4 Compared with L–K functionals in many present works, it is easily checked that the L–K functionals proposed in [2,5,6,8,12,13,21–24,27,32,34,41,43–46] individually employ the information of neutral delay $\tau_1(t)$ and state one $\tau_2(t)$. However, in our work, the multiple integral terms in $V_4(x_t)$ constructed in (15) can reflect the interconnection between two kinds of time-delays $\tau_i(t)$ $(i = 1, 2)$ as much as possible, which results in that Theorems 1–2 not only depend on τ_i $(i = 1, 2)$ but also on the values $\tau_{21}, \delta_{21}, \theta_{21}$ in (4). Thus, these L–K functional terms can play an important role in reducing the conservatism effectively when $\tau_1(t) \neq \tau_2(t)$.

Remark 5 As illustrated in Ref. [46], the general form of modeling partial element equivalent circuit (PEEC) can be modeled as

$$\begin{aligned}&C_0\dot{y}(t) + G_0 y(t) + C_1\dot{y}(t-\tau) + G_1 y(t-\tau) = Bu(t, t-\tau), \quad t \geq t_0;\\ &y(t) = \phi(t), \quad t \leq t_0.\end{aligned} \tag{39}$$

To be consistent with the mathematical deduction, the system (39) can be rewritten as the following neutral system

$$\begin{aligned}&\dot{y}(t) = Ay(t) + By(t-\tau) + C\dot{y}(t-\tau), \quad t \geq t_0;\\ &y(t) = \phi(t), \quad t \leq t_0.\end{aligned} \tag{40}$$

As we know, a stable numerical solution should be based on a stable model. Therefore, the study of asymptotic stability of a system is an important issue before handling its numerical solution. Thus, the delay-dependent stability of system (40) was investigated in some existent works. Thus, if we take the parameter uncertainties commonly existing in the modeling of a real circuit and different time-varying delays into account, a more general form of (40) can be described by the following system

$$\begin{aligned}&\dot{y}(t) = [A + \Delta A(t)]y(t) + [B + \Delta B(t)]y(t-\tau_2(t)) + C\dot{y}(t-\tau_1(t)), \quad t \geq t_0;\\ &y(t) = \phi(t), \quad t \leq t_0.\end{aligned} \tag{41}$$

Therefore, the derived theorems in this work can be applied to study the stability for the PEEC model with more general forms.

Remark 6 If there exist the multiple time-varying delays in the state of the system (1), i.e.,

$$\begin{aligned}&\dot{x}(t) - [C + \Delta C(t)]\dot{x}(t-\tau_1(t)) = [A + \Delta A(t)]x(t)\\ &\quad + \sum_{j=2}^{l}[B_i + \Delta B_i(t)]x(t-\tau_i(t)), \quad t \geq t_0; \quad x(t) = \phi(t), \quad t \leq t_0,\end{aligned} \tag{42}$$

where the time-delays satisfies

$$0 \leq \tau_i(t) \leq \tau_i, \quad \upsilon_i \leq \dot{\tau}_i(t) \leq \mu_i \quad (i = 1, 2, \ldots, l). \tag{43}$$

Then, if we denote $\tau_{ji} = \tau_j - \tau_i$, $\delta_{ji} = \tau_j^2 - \tau_i^2$, and $\theta_{ji} = \tau_j^3 - \tau_i^3$ $(i \neq j)$, we will construct the following Lyapunov terms to express the interconnection between $\tau_i(t)$ and $\tau_j(t)$ $(i, j = 1, \ldots, l)$

$$\sum_{1\le i\ne j\le l}^{l} \tau_{ji} \int_{t-\tau_j}^{t-\tau_i} x^{\mathrm{T}}(s)U_i x(s)\mathrm{d}s,$$

$$\sum_{1\le i\ne j\le l}^{l} \frac{\delta_{ji}}{2} \int_{-\tau_j}^{-\tau_i} \int_{\varrho}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)X_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho, \tag{44}$$

$$\sum_{1\le i\ne j\le l}^{l} \tau_{ji} \int_{t-\tau_j}^{t-\tau_i} \int_{t+s}^{t} \left[\dot{x}^{\mathrm{T}}(\theta)V_i\dot{x}(\theta) + x^{\mathrm{T}}(\theta)W_i x(\theta)\right]\mathrm{d}\theta\mathrm{d}s, \tag{45}$$

$$\sum_{1\le i\ne j\le l}^{l} \frac{\theta_{ji}}{6} \int_{-\tau_j}^{-\tau_i} \int_{\mu}^{0} \int_{\varrho}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Y_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho\mathrm{d}\mu. \tag{46}$$

4 Numerical Examples

In this section, two numerical examples will be presented to illustrate the derived results.

Example 1 Consider the partial element equivalent circuit (PEEC) model (41) with the parameters [19,46]

$$A = 100 \times \begin{bmatrix} -2.105 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix},\ B = 100 \times \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$C = \frac{1}{72} \times \begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix};$$

$$\| \Delta A(t) \| \le 2, \quad \| \Delta B(t) \| \le 2, \quad \| \Delta C(t) \| = 0.$$

Then, we can assume that $F = \mathrm{diag}\{1, 1, 1\}$, $E_1 = E_2 = \mathrm{diag}\{2, 2, 2\}$, and $J = E_3 = 0_{3\times 3}$.

Firstly, as for the case $\tau_1(t) = \tau_2(t)$, with the existent feasible solution to the LMIs in Theorem 2, the computational results on maximum allowable upper bounds (MAUBs) of time-delays for various υ_1, μ_1 can be computed out, and meanwhile, we can obtain the corresponding MAUBs based on the theorems in [8,16,33]. Together with all derived MAUBs listed in Table 1, one can check that Theorem 2 can be superior over some present ones. Since restricted conditions on the theorems are required in [8,16,33], during the computing, we assume that the nonlinear function does not exist with $\alpha = 0.5$ in [8], $\rho = 0.5$ is given in [16], and the lower bound of time-delay is set as 0 in [33]. It is worth noting that since the upper bound of neutral delay's derivative has to be less than 1, there does not exist the corresponding MAUBs as $\mu_1 > 1$ in [8,16,33].

Secondly, as for $\tau_1(t) \ne \tau_2(t)$, based on the conditions [8,16,33], we can choose $\tau_1(t) = 0.3\sin^2(t)$. Then, $\tau_1 = 0.3$ and $\upsilon_1 = -0.3$, $\mu_1 = 0.3$. In what follows, through setting different υ_2, μ_2, we also can derive the corresponding MAUBs of

Table 1 Calculated MAUBs $\tau_{\max}$ for different υ_1, μ_1, and $\tau_1(t) = \tau_2(t)$

μ_1	0.4	0.7	1.1
Ren et al. [33]	0.3654	0.2434	–
Liu [16]	0.3686	0.2465	–
Duan et al. [8]	0.3435	0.2264	–
Theorem 2 ($\upsilon_1 = -1.1$)	0.3732	0.2528	0.2522
Theorem 2 ($\upsilon_1 = -0.5$)	0.3743	0.2543	0.2540
Theorem 2 ($\upsilon_1 = -0.2$)	0.3749	0.2556	0.2553

Table 2 Calculated MAUBs $\tau_{\max}$ for different υ_2, μ_2, and $\tau_1(t) \neq \tau_2(t)$

μ_2	0.4	0.7	1.1
Ren et al [33]	0.3654	0.2434	0.2302
Liu [16]	0.3686	0.2465	0.2335
Duan et al [8]	0.3435	0.2264	0.2138
Theorem 2 ($\upsilon_2 = -1.1$)	0.3785	0.2562	0.2544
Theorem 2 ($\upsilon_2 = -0.5$)	0.3792	0.2580	0.2572
Theorem 2 ($\upsilon_2 = -0.2$)	0.3795	0.2594	0.2588

τ_2 based on the LMIs in Theorem 2 and the results in [8,16,33], which are listed in Table 2. From Tables 1, 2 and 3, one can check that our method is less conservative than those existent ones. Therefore, it is of significance to use the interconnection between the neutral delay and state one to construct the Lyapunov functional, especially when neutral delay and sate one are different.

Thirdly, similar to Refs. [19,46], we choose the matrix $A = 100 \times \begin{bmatrix} \delta & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}$ with $\delta < 0$. In particular, in [14], the nonlinearities can be equivalently converted into the normal uncertainties, i.e.,

$$\begin{aligned} f_1\big(t, x(t)\big) &\doteq \Delta A(t)x(t),\ f_2\big(t, x(t-\tau(t))\big) \\ &\doteq \Delta B(t)x(t-\tau(t)),\ f_3\big(t, \dot{x}(t-\tau(t))\big) \doteq \Delta C(t)\dot{x}(t-\tau(t)). \end{aligned}$$

In what follows, in order to give better comparison, we let $\| \Delta A(t) \| = \| \Delta B(t) \| \leq 0.01$ and $\| \Delta C(t) \| = 0$. Since in [19,46], the upper bound of $\dot{\tau}(t)$ should be less than 1, and thus, we choose $\tau(t) = 0.1 + 0.3\sin^2(0.2t)$. Based on Theorem 2 and using MATLAB LMI Toolbox, it can be shown that the system is robustly stable for $\delta \leq -3.445$. However, since $\delta \leq -3.810$ in [19] and $\delta \leq -4.465$ in [46] are needed, these criteria fail to check the robust stability with $\delta = -3.5$.

Example 2 In the example, we consider the system (1) with the following parameters

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 & 0 \\ 0.5 & -0.5 \end{bmatrix},\ C = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix},$$

Table 3 Calculated MAUBs of τ_2 for various υ_2, μ_2, and $\tau_1(t) = 1.2\sin^2(0.5t)$

μ_2	0.5	0.9	1.1
Lu et al. [18]	1.5572	1.5572	1.5572
Wang et al. [41]	1.6635	1.5742	1.5645
Liu et al. [24]	1.5812	1.5745	1.5644
Theorem 2 ($\upsilon_2 = -1.1$)	1.8675	1.7244	1.7194
Theorem 2 ($\upsilon_2 = -0.5$)	1.8763	1.7352	1.7333
Theorem 2 ($\upsilon_2 = -0.2$)	1.8780	1.7453	1.7445

and the uncertainties $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$ are presented in (3) with

$$\begin{aligned} F &= \mathrm{diag}\{1, 1\}, \quad E_1 = \mathrm{diag}\{0.05, 0.05\}, \\ E_2 &= \mathrm{diag}\{0.1, 0.1\}, \quad E_3 = J = 0_{2\times 2}. \end{aligned}$$

Now together with the theorems in [18,24,41] and Theorem 2 in this work, the purpose of this example is also to compute out the MAUBs on time-delays and give some comparing results among them. Here, we mainly aim to the case $\tau_1(t) \neq \tau_2(t)$. Firstly, we set $\tau_1(t) = 1.2\sin^2(0.5t)$, then $\tau_1 = 1.2$, $\upsilon_1 = -0.6$, $\mu_1 = 0.6$. Now based on various υ_2, μ_2, we compute out corresponding MAUBs of τ_2. During the discussion, since there does not include the information on delay's derivative in [18], the derived MAUBs are identical during adjusting υ_2, μ_2. In Table 3, owing to that the L–K functional and Theorem 2 can represent the interconnection of neutral delay and state one, thus, the application area of our results can be greatly extended.

5 Conclusions

In this work, one novel mixed-delay-dependent condition on robust stability has been established for a class of uncertain neutral systems with time-varying delays. Compared with some existing results, the derived results are mainly based on an augmented L–K functional and can effectively reduce the conservatism owing to using some effective techniques, in which the interconnection between the neutral delay and state one has been deeply studied. Moreover, together with the utilization of some novel integral inequalities, those previously ignored information has been effectively reconsidered in this paper. Two numerical examples show the benefits of the proposed techniques and its application to PEEC model.

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Appendix

In what follows, some lemmas will be presented for the proof procedure of Theorems 1–2.

Lemma 1 [50] *For $d(t) \in [0, d]$, a symmetric matrix $R > 0$ and any matrix S_1 satisfying $\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0$ with $R_1 = \mathrm{diag}\{R, 3R, 5R\}$, the following inequality can be true*

$$
\begin{aligned}
&-\int_{t-d(t)}^{t} \dot{x}(s)ds - \int_{t-d}^{t-d(t)} \dot{x}(s)ds \\
&\leq -\frac{1}{d}\zeta^{\mathrm{T}}(t)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} + \begin{bmatrix} \frac{d-d(t)}{d}T & 0 \\ * & \frac{d(t)}{d}T \end{bmatrix} \right) \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \zeta(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta^{\mathrm{T}}(t) &= \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-d(t)) & x^{\mathrm{T}}(t-d) & \varphi^{\mathrm{T}}(t) & \varrho^{\mathrm{T}}(t) & \nu^{\mathrm{T}}(t) & \omega^{\mathrm{T}}(t) \end{bmatrix};\\
E_1 &= \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_4 \\ e_1 - e_2 + 6e_4 - 12e_6 \end{bmatrix}, \quad E_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_5 \\ e_2 - e_3 + 6e_5 - 12e_7 \end{bmatrix};\\
e_i &= \begin{bmatrix} 0_{i-1} & I_n & 0_{7-i} \end{bmatrix} (1 \leq i \leq 7), \quad T = R_1 - S_1^{\mathrm{T}} R_1^{-1} S_1;\\
\varphi(t) &= \frac{1}{d(t)} \int_{t-d(t)}^{t} x(s)\mathrm{d}s, \quad \nu(t) = \frac{2}{d^2(t)} \int_{t-d(t)}^{t} \int_{t-d(t)}^{s} x(u)\mathrm{d}u\mathrm{d}s;\\
\varrho(t) &= \frac{1}{d-d(t)} \int_{t-d}^{t-d(t)} x(s)\mathrm{d}s, \quad \omega(t) = \frac{2}{(d-d(t))^2} \int_{t-d}^{t-d(t)} \int_{t-d}^{s} x(u)\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

Lemma 2 [28,50] *For an any constant matrix $M > 0$, the following inequalities hold for all continuously differentiable function φ in $[a, b] \to \mathbf{R}^n$:*

$$
\begin{aligned}
&-(b-a)\int_a^b \varphi^{\mathrm{T}}(s)M\varphi(s)ds \leq -\left(\int_a^b \varphi(s)ds\right)^{\mathrm{T}} M \left(\int_a^b \varphi(s)ds\right) - 3\Theta^{\mathrm{T}}M\Theta,\\
&-\frac{b^2-a^2}{2}\int_a^b \int_{t+\theta}^{t} \varphi^{\mathrm{T}}(s)M\varphi(s)dsd\theta\\
&\leq -\left(\int_a^b \int_{t+\theta}^{t} \varphi(s)dsd\theta\right)^{\mathrm{T}} M \left(\int_a^b \int_{t+\theta}^{t} \varphi(s)dsd\theta\right),\\
&-\frac{b^3-a^3}{6}\int_{-b}^{-a} \int_{\varrho}^{0} \int_{t+\theta}^{t} \varphi^{\mathrm{T}}(s)M\varphi(s)dsd\theta d\varrho\\
&\leq -\left(\int_{-b}^{-a} \int_{\varrho}^{0} \int_{t+\theta}^{t} \varphi(s)dsd\theta d\varrho\right)^{\mathrm{T}} M \left(\int_{-b}^{-a} \int_{\varrho}^{0} \int_{t+\theta}^{t} \varphi(s)dsd\theta d\varrho\right),
\end{aligned}
$$

where $\Theta = \int_a^b \varphi(s)ds - \frac{2}{b-a}\int_a^b \int_a^s \varphi(u)duds$.

Lemma 3 [28] *For an any constant matrix* $M > 0$*, the following inequality holds for all continuously differentiable function* φ *in* $[a, b] \to \mathbf{R}^n$*:*

$$\begin{aligned}&-\frac{(b-a)^2}{2}\int_a^b \int_a^s \varphi^{\mathrm{T}}(u)M\varphi(u)duds\\&\le -\left(\int_a^b \int_a^s \varphi(u)duds\right)^{\mathrm{T}} M \left(\int_a^b \int_a^s \varphi(u)duds\right) - 2\Theta^{\mathrm{T}}M\Theta,\end{aligned}$$

where $\Theta = \int_a^b \int_a^s \varphi(u)duds - \frac{3}{b-a}\int_a^b \int_a^s \int_a^u \varphi(v)dvduds$.

Lemma 4 [29] *For vector* ω*, real scalars* $a \le b$*, symmetric matrix* $R > 0$ *such that the integration is well defined, then the following inequality holds,*

$$(b-a)\int_a^b \dot{\omega}^{\mathrm{T}}(s)R\dot{\omega}(s)ds \ge \chi_1^{\mathrm{T}}R\chi_1 + 3\chi_2^{\mathrm{T}}R\chi_2 + 5\chi_3^{\mathrm{T}}R\chi_3,$$

where

$$\begin{aligned}&\chi_1 = \omega(b) - \omega(a), \quad \chi_2 = \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(s)ds,\\&\chi_3 = \omega(b) - \omega(a) + \frac{6}{b-a}\int_a^b \omega(s)ds - \frac{12}{(b-a)^2}\int_a^b \int_s^b \omega(\theta)d\theta ds.\end{aligned}$$

As an extended case of Lemma 2 in [25], we can derive the following Lemma easily.

Lemma 5 [25] *Suppose that* Ω, Ξ_{ij}, Ξ_{mn} $(i, m = 1, 2, 3, 4;\ j, n = 1, 2)$ *are the constant matrices of appropriate dimensions,* $\alpha \in [0, 1]$, $\beta \in [0, 1]$, $\gamma \in [0, 1]$*, and* $\delta \in [0, 1]$*, then*

$$\begin{aligned}&\Omega + \left[\alpha\Xi_{11} + (1-\alpha)\Xi_{12}\right] + \left[\beta\Xi_{21} + (1-\beta)\Xi_{22}\right]\\&+\left[\gamma\Xi_{31} + (1-\gamma)\Xi_{32}\right] + \left[\delta\Xi_{41} + (1-\delta)\Xi_{42}\right] < 0\end{aligned}$$

holds, if and only if the following inequalities hold simultaneously,

$$\Omega + \Xi_{ij} + \Xi_{mn} < 0 \ \ (i, m = 1, 2, 3, 4;\ j, n = 1, 2).$$

Lemma 6 [26,38] *Let* $I - G^{\mathrm{T}}G > 0$ *define the set* $\Upsilon = \left\{\Delta(t) = \Sigma(t)[I - G\Sigma(t)]^{-1}, \Sigma^{\mathrm{T}}(t)\Sigma(t) \le I\right\}$*, for given matrices* H, J*, and* R *of appropriate dimensions and symmetric one* H*, then* $H + J\Delta(t)R + R^{\mathrm{T}}\Delta^{\mathrm{T}}(t)J^{\mathrm{T}} < 0$*, iff there exists* $\rho > 0$ *such that*

$$H + \begin{bmatrix} \rho^{-1} R \\ \rho J^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & -G \\ -G^{\mathrm{T}} & I \end{bmatrix}^{-1} \begin{bmatrix} \rho^{-1} R \\ \rho J^{\mathrm{T}} \end{bmatrix} < 0.$$

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